The Next "Free Lunches"

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SUMMARY

This article introduces two generations of "free lunches" for long-term investors. These "free lunches" expand the classic Modern Portfolio Theory based "free lunch" to the areas of glide path design and funding problem optimization. These "free lunches" are applicable to defined contribution, defined benefit, and college savings plans, as well as foundations, endowments, and other segments of institutional and individual investors. This author is optimistic that these new generations of "free lunches" will become broadly accepted concepts in investing.

"Diversification is the only free lunch" is one of the most renowned maxims in finance. This "free lunch" constitutes the ability to eliminate certain inefficiencies in a single-period portfolio selection. Modern Portfolio Theory (MPT) provides the conceptual and analytical tools for the portfolio selection "free lunch."

The notion that a great value could be created at little cost is hard to find in investing. Hence, numerous publications have discussed and challenged the concept of "free lunch" from different angles. This article adds another one – it challenges the "only" part of the maxim by means of introducing two new generations of "free lunches."

As demonstrated in this article, the MPT "free lunch" – called *"Free Lunch" 1.0* – is essentially an outcome optimizing methodology for a single-period funding problem. Yet there is a multitude of investors with multi-period funding problems (e.g. retirement investors) that require optimizing series of portfolios (a.k.a. glide paths) that extend throughout the investor's time horizon. Just like portfolios, glide paths can be efficient and inefficient.

This article introduces a "free lunch" for optimal glide path design; we call this approach *"Free Lunch" 2.0*. This "free lunch" constitutes the ability to eliminate certain inefficiencies in *portfolio selection and evolution*. This "free lunch" for glide paths is remarkably similar to its classic MPT based counterpart.

Yet an investor with a funding problem may need much more than optimal glide path design and its "free lunch." Such an investor may require optimizing not only his glide path, but the funding problem itself. For example, investors may want to maximize the feasible commitment out-flows and/or minimize the required commitment in-flows and/or minimize the overall riskiness of the funding problem. Each of these objectives may have its own "free lunch" that comes with its own optimal glide path. The framework that makes the funding problem "free lunches" possible is called *"Free Lunch" 3.0* in this article.

Overall, this article introduces two generations of "free lunches" for investors with long-term funding problems. These "free lunches" represent natural extensions of the classic MPT based "free lunch." These "free lunches" are tangible and substantial; they reflect the existence of multiple investment objectives; they can be tailored to different groups of stakeholders (e.g. plan participants, taxpayers, shareholders); they can be expressed in terms of "implied surcharges," contributions, and payments. These "free lunches" are applicable to defined contribution, defined benefit, and college savings plans as well as foundations, endowments, and other investors. This author is optimistic that these generations of "free lunches" will become mainstream concepts in investing.

"Free Lunch" 1.0: Modern Portfolio Theory

A *funding problem* represents a set of commitments to make future contributions and payments. The outcomes of a typical funding problem may be uncertain due to the fact that many investors employ risky assets to fund their financial commitments. Thus, a funding problem may include commitment in-flows (contributions), out-flows (payments), as well as risk measurements and their evolution. (Note that the existing assets are considered an in-flow.) We assume that the sole purpose of commitment in-flows is to fund commitment out-flows.

Let us consider the following funding problem: given $1 now, pay $1 at the end of the year. The outcome of this single-period funding problem is the remaining asset value after paying $1 at the end of the year. We call this value *terminal asset value* (*TAV*).

This funding problem is directly related to MPT. To establish this relationship, note that the accumulated value of today's $1 is $1 + R$, where R is portfolio return; *TAV* is equal to the accumulated value minus $1: $TAV = 1 + R - 1 = R$; thus, *portfolio return optimization and TAV optimization are one and the same*.

The next step is to define investment objectives. As stated in Markowitz [1952], "*We next consider the rule that the investor does or should consider expected return a desirable thing and variance of return an undesirable thing.*" Hence, the investor's primary objective is dual: to maximize the expected return (*TAV*) and to minimize the volatility of return (*TAV*). Clearly, *TAV* is the key outcome variable and expected *TAV* and volatility of *TAV* are the vital outcome measurements. Thus, *TAV* in a single-period funding problem is at the heart of MPT.

One way to generate optimal portfolios is to consider the following objectives:

- to maximize the expected *TAV* given volatility of *TAV*;
- to minimize the volatility of *TAV* given expected *TAV*.

These objectives lead to the same set of optimal portfolios called *efficient frontier*.

Another way to generate optimal portfolios is to define "the risk-adjusted expected *TAV*" (*RAETAV*) $V_X = E_X - tS_X$, where X is a portfolio, E_X is the expected *TAV*, S_X is the standard deviation of *TAV*, $t \geq 0$ is a risk aversion factor. Maximizing V_X for all $t \geq 0$ generates the efficient frontier.

The rest can be found in most finance textbooks.

"Free Lunch" 2.0: Glide Path Optimization

As demonstrated in in the previous section, MPT essentially seeks to optimize the outcomes of a simple single-period funding problem. Obviously, this funding problem is just one of a multitude of short- and long-term funding problems. To illustrate the concept of "free lunch" for long-term investors, this section presents a simplified two-period funding problem.

Let us consider the following funding problem: given $2 now, pay $1 at the end of the first year and $1 at the end of the second year. There are two portfolio selections to be made: at the present and at the beginning of the second year.

We distinguish two types of glide paths as related to these portfolio selections. The first type assumes that the investor will follow the glide path designed today. We call such glide paths "*will-do*" glide paths. The second type is based on rational expectations of the investor's future portfolio selections. It assumes that the investor will optimize remaining (sub)-glide paths at the future portfolio selection points. We call such glide paths "*expected-to-do*" glide paths. In this section, we examine "will-do" glide paths and "expected-to-do" glide paths separately.

For simplicity, let us assume the investor has decided to utilize stock and bond index funds only. Consequently, a portfolio's equity allocation identifies the portfolio; every portfolio is MPT-efficient. The capital market assumptions are presented in Appendix II.

Let us apply the methodology of the previous section to "will-do" glide paths. Specifically, let us define "the risk-adjusted expected *TAV*" (*RAETAV*) $V_{X,Y} = E_{X,Y} - tS_{X,Y}$, where X is a portfolio selected at the present, Y is a portfolio selected at the beginning of the second year, *TAV* is the terminal asset value after making all payments at the end of the second year, $E_{X,Y}$ is the expected *TAV*, $S_{X,Y}$ is the standard deviation of *TAV*, $t \geq 0$ is a risk aversion factor. Maximizing $V_{X,Y}$ for all $t \geq 0$ generates a set of efficient two-portfolio glide paths. For the reader's convenience, the formulas for expected *TAV* and standard deviation of *TAV* are presented in *Appendix I*.

Exhibit 1 contains the "will-do" efficient glide paths for the selected risk aversion factors from 0.30 to 3.00.

Exhibit 1

"Will-Do" Optimal Glide Paths											
Risk Aversion *t*	0.30	0.35	0.40	0.45	0.50	0.60	0.80	1.00	1.50	2.00	3.00
Equity Year 1	100.0%	70.1%	54.0%	39.2%	32.0%	24.6%	18.0%	14.8%	11.1%	9.3%	7.6%
Equity Year 2	100.0%	100.0%	97.7%	69.5%	55.8%	41.6%	29.1%	23.1%	15.9%	12.6%	9.3%
Expected *TAV*	0.2537	0.2285	0.2138	0.1891	0.1771	0.1649	0.1541	0.1489	0.1428	0.1399	0.1372
St Dev *TAV*	0.3706	0.2929	0.2537	0.1951	0.1697	0.1471	0.1313	0.1254	0.1203	0.1186	0.1175
Optimal *RAETAV*	0.1425	0.1260	0.1123	0.1013	0.0923	0.0766	0.0491	0.0235	-0.0377	-0.0973	-0.2153

Here is how to read *Exhibit 1*. The optimal "will-do" glide path for risk aversion factor 0.5 consists of portfolios 32.0/68.0 (% stocks/bonds) in the first year and 55.8/44.2 in the second year. The mean and standard deviation of *TAV* are 0.1771 and 0.1697 correspondingly. The optimal value of the risk-adjusted expected *TAV* is 0.0923.

Most glide paths currently offered by target date funds (TDF) are (partially) stationary in retirement. Yet, the optimal glide paths in *Exhibit 1* are generally evolving, which implies that stationary glide paths are inefficient. How significant is this inefficiency?

To answer this question, we take the following two steps. First, for every optimal glide path in *Exhibit 1*, we find a stationary glide path that has the same standard deviation of *TAV*. Second, we measure the inefficiency of this stationary glide path as the "implied surcharge" – the reduction in expected return required to make the optimal glide path's mean of *TAV* equal of its counterpart for the stationary glide path with unreduced returns. *Exhibit 2* contains the results.

Here is how to read *Exhibit 2*. The optimal "will-do" glide path for risk aversion factor 0.5 consists of portfolios 32.0/68.0 (% stocks/bonds) in the first year and 55.8/44.2 in the second year. The mean and standard deviation of *TAV* are 0.1771 and 0.1697 correspondingly. The mean and standard deviation of *TAV* for the stationary glide path 38.8/61.2 in both years are 0.1754 and 0.1697 correspondingly. The optimal value of the risk-adjusted expected *TAV* is 0.0923. The expected return reduction required to reduce the optimal glide path's expected *TAV* from 0.1771 to 0.1754 is equal to 5.4 basis points. *Exhibit 2* demonstrates that there are tangible "surcharges" imposed by the inefficiencies of stationary glide paths.

Exhibit 2

"Will-Do" Optimal Glide Paths											
Risk Aversion *t*	0.30	0.35	0.40	0.45	0.50	0.60	0.80	1.00	1.50	2.00	3.00
Equity Year 1	100.0%	70.1%	54.0%	39.2%	32.0%	24.6%	18.0%	14.8%	11.1%	9.3%	7.6%
Equity Year 2	100.0%	100.0%	97.7%	69.5%	55.8%	41.6%	29.1%	23.1%	15.9%	12.6%	9.3%
Expected *TAV*	0.2537	0.2285	0.2138	0.1891	0.1771	0.1649	0.1541	0.1489	0.1428	0.1399	0.1372
St Dev *TAV*	0.3706	0.2929	0.2537	0.1951	0.1697	0.1471	0.1313	0.1254	0.1203	0.1186	0.1175
Optimal *RAETAV*	0.1425	0.1260	0.1123	0.1013	0.0923	0.0766	0.0491	0.0235	-0.0377	-0.0973	-0.2153
Stationary Glide Path											
Equity Year 1	100.0%	78.2%	66.6%	47.9%	38.8%	29.4%	21.2%	17.2%	12.5%	10.3%	8.2%
Equity Year 2	100.0%	78.2%	66.6%	47.9%	38.8%	29.4%	21.2%	17.2%	12.5%	10.3%	8.2%
Expected *TAV*	0.2537	0.2256	0.2107	0.1869	0.1754	0.1636	0.1533	0.1483	0.1425	0.1398	0.1371
St Dev *TAV*	0.3706	0.2929	0.2537	0.1951	0.1697	0.1471	0.1313	0.1254	0.1203	0.1186	0.1175
"Surcharge" (bp)	0.0	8.8	9.3	6.7	5.4	3.8	2.5	1.8	1.0	0.6	0.2

Now let us apply the methodology of the previous section to "expected-to-do" glide paths. We assume that the investor will exercise his risk tolerance and optimize remaining (sub)-glide paths at the future portfolio selection points. For simplicity, let us assume that the investor's risk aversion factor remains the same $t \geq 0$ throughout the investor's time horizon.

For the first year, optimal portfolio X is selected via maximizing the risk-adjusted expected *TAV* $V_X = E_X - tS_X$, where E_X is the expected *TAV* at the end of the first year, S_X is the standard deviation of *TAV* at the end of the first year.

For the second year, optimal portfolio Y is selected via maximizing the risk-adjusted expected *TAV* $V_{X,Y} = E_{X,Y} - tS_{X,Y}$, where X is the portfolio selected for the first year, $E_{X,Y}$ is expected *TAV* the end of the second year, $S_{X,Y}$ is standard deviation of *TAV* the end of the second year. For every optimal "expected-to-do" glide path, we also calculate a stationary glide path that has the same standard deviation of *TAV*, as we just did for "will-do" glide paths. *Exhibit 3* contains the results.

Exhibit 3

"Expected-To-Do" Optimal Glide Paths, Constant Risk Aversion Factor											
Risk Aversion *t*	0.30	0.35	0.40	0.45	0.50	0.60	0.80	1.00	1.50	2.00	3.00
Equity Year 1	65.8%	42.7%	33.8%	28.7%	25.2%	20.9%	16.2%	13.7%	10.5%	9.0%	7.5%
Equity Year 2	100.0%	100.0%	79.1%	62.5%	52.3%	40.4%	28.8%	23.0%	15.9%	12.6%	9.3%
Expected *TAV*	0.2248	0.2053	0.1887	0.1773	0.1701	0.1613	0.1525	0.1479	0.1423	0.1396	0.1371
St Dev *TAV*	0.2826	0.2344	0.1955	0.1706	0.1565	0.1415	0.1294	0.1245	0.1200	0.1185	0.1174
Optimal *RAETAV*	0.1400	0.1233	0.1105	0.1005	0.0918	0.0764	0.0490	0.0234	-0.0377	-0.0973	-0.2153
Stationary Glide Path											
Equity Year 1	75.2%	60.7%	48.1%	39.1%	33.5%	26.8%	20.0%	16.4%	12.1%	10.1%	8.1%
Equity Year 2	75.2%	60.7%	48.1%	39.1%	33.5%	26.8%	20.0%	16.4%	12.1%	10.1%	8.1%
Expected *TAV*	0.2217	0.2032	0.1871	0.1758	0.1688	0.1603	0.1518	0.1474	0.1420	0.1395	0.1370
St Dev *TAV*	0.2826	0.2344	0.1955	0.1706	0.1565	0.1415	0.1294	0.1245	0.1200	0.1185	0.1174
"Surcharge" (bp)	9.4	6.6	5.0	4.4	3.9	3.1	2.1	1.6	0.9	0.5	0.2

Here is how to read *Exhibit 3*. The optimal "expected-to-do" glide path for risk aversion factor 0.5 consists of portfolios 25.2/74.8 (% stocks/bonds) in the first year and 52.3/47.7 in the second year. The mean and standard deviation of *TAV* are 0.1701 and 0.1565 correspondingly. The optimal value of the risk-adjusted expected *TAV* is 0.0918. The mean and standard deviation of *TAV* for the stationary glide path 33.5/66.5 in both years are 0.1688 and 0.1565 correspondingly. The expected return reduction required to reduce the optimal glide path's expected *TAV* from 0.1701 to 0.1688 is equal to 3.9 basis points. *Exhibit 3* demonstrates that there are tangible "surcharges" imposed by the inefficiencies of stationary glide paths.

The "surcharges" presented in *Exhibit 2* and *Exhibit 3* represent the "free lunch" that comes from optimal portfolio selection and evolution for "will-do" and "expected-to-do" glide paths.

"Free Lunch" 3.0: Funding Problem Optimization

In the previous sections, we optimized portfolio selection and evolution while keeping the underlying funding problem unaffected. In this section, we take the next step – to optimize not only glide paths, but also the key elements of the funding problem. These key elements include, but are not limited to, the funding problem's cash flows – commitment in-flows and out-flows.

The first step in the optimization of a funding problem is to specify investment objectives. Loosely speaking, the investor's primary objective is to fund the commitment out-flows given the commitment in-flows. The appropriate levels of commitment in-flows and out-flows are yet to be determined as they depend on the nature of the investor and his attitude toward risk.

For example, the commitment out-flows (benefit payments) for a DB plan can be considered given. The plan determines the appropriate levels of commitment in-flows (subject to relevant regulations). In general, lower required contributions to fund given benefits are considered desirable.

As another example, the commitment in-flows (saving rates) for a DC plan participant can be considered given. The appropriate levels of commitment out-flows (retirement spending) are yet to be determined. In general, higher retirement spending given saving rates is considered desirable.

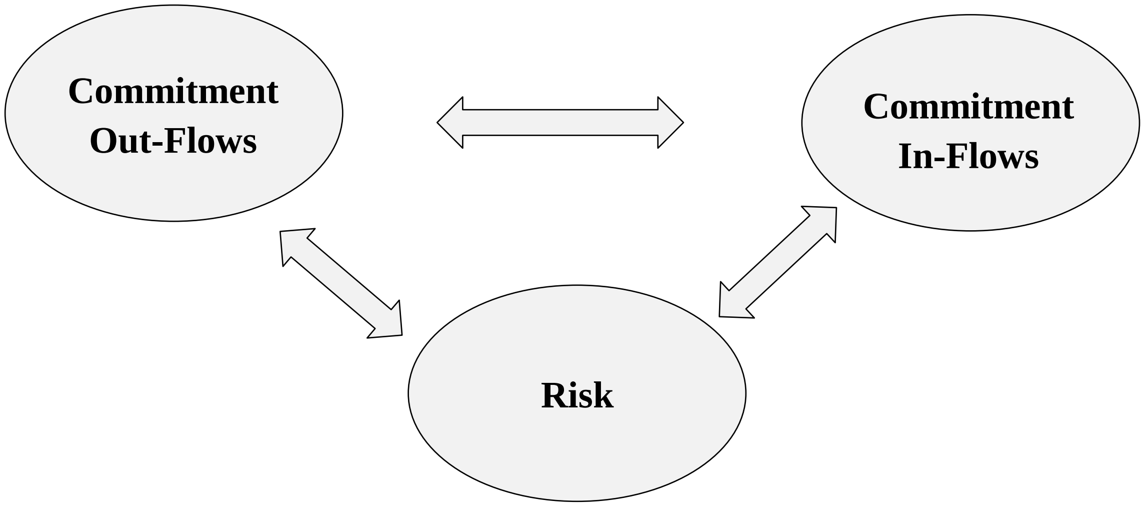
Due to the use of risky assets in the funding process, the investor may or may not succeed in achieving the primary objective. Hence, *the primary risk is defined as the failure of the primary objective*. In other words, the primary risk is defined as the shortfall event – the commitment in-flows are insufficient to fund the commitment out-flows.

To manage the primary risk, we need to measure it. The principle of "time value of money" requires measuring all cash flows at the same point in time. In this paper, we choose the end of the time horizon as the measurement point. Risk measurements include, but are not limited to, shortfall probability, size, and volatility. To be consistent with the rest of this paper, we select the risk-adjusted expected *TAV* as the primary measurement.

It is important to note that *the risk-adjusted expected TAV should be equal to zero* since the sole purpose of commitment in-flows is to fund commitment out-flows. In other words, we "expect" neither a *TAV* "surplus" nor a *TAV* "shortfall." Commitment in- and out-flows should be adjusted accordingly.

It is convenient to depict the key elements of the funding problem and their relationships ("*the funding triangle*") graphically, see *Exhibit 4*. Note that the term "Risk" in the funding triangle should be understood as a "shortcut" for "the selected measurement of the primary risk."

Exhibit 4 **The Funding Triangle**



The concept of the funding triangle allows to define the investment objectives precisely and concisely. The investment objectives are defined according to the following principle:

Given two vertices of the funding triangle, optimize the third.

Specifically, there are three distinct investment objectives:

A. Given commitment in- and out-flows, minimize risk.
B. Given commitment in-flows and risk, maximize commitment out-flows.
C. Given commitment out-flows and risk, minimize commitment in-flows.

As an example, let us consider objective A for the funding problem from the previous section: given \$2 now, pay \$1 at the end of the first year and \$1 at the end of the second year. For this funding problem,

- the commitment in-flow is \$2 contributed at the present;
- the commitment out-flows are a \$1 payment at the end of the first year and a \$1 payment at the end of the second year;
- the objective is to maximize the risk aversion factor.

Utilizing "expected-to-do" glide paths, the highest risk aversion factor is 1.189 generated by the optimal "expected-to-do" glide path 12.1/87.9 (% stocks/bonds) in the first year and 19.5/80.5 in the second year. The mean and standard deviation of *TAV* are 0.1452 and 0.1221 correspondingly. The optimal value of the risk-adjusted expected *TAV* is zero, as intended.

Objective B implies the following funding problem: given \$2 today and a risk aversion factor, maximize the (same) payments at the end of the first and the second years. For this funding problem,

- the commitment in-flow is \$2 contributed at the present;
- the commitment out-flows are an \$*X* payment at the end of the first year and an \$*X* payment at the end of the second year;
- the objective is to maximize payments \$*X* given a risk aversion factor.

The results for various risk aversion factors are presented in *Exhibit 5*.

Exhibit 5

"Expected-To-Do" Optimal Glide Paths: \$2 Asset Value, Maximum Payments											
Risk Aversion *t*	0.30	0.35	0.40	0.45	0.50	0.60	0.80	1.00	1.50	2.00	3.00
Equity Year 1	65.8%	42.7%	33.8%	28.7%	25.2%	20.9%	16.2%	13.7%	10.5%	9.0%	7.5%
Equity Year 2	100.0%	100.0%	82.1%	64.5%	53.9%	41.3%	29.2%	23.1%	15.8%	12.4%	9.2%
Expected *TAV*	0.0830	0.0798	0.0776	0.0762	0.0777	0.0844	0.1031	0.1242	0.1807	0.2393	0.3602
St Dev *TAV*	0.2766	0.2280	0.1941	0.1694	0.1555	0.1406	0.1288	0.1242	0.1204	0.1196	0.1200
Max Payments	1.068	1.060	1.054	1.049	1.045	1.038	1.024	1.012	0.981	0.951	0.891
Optimal *RAETAV*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Here is how to read *Exhibit 5*. For risk aversion factor 0.5, the highest payment amount is \$1.045 (payable at the end of the first and the second year). The optimal "expected-to-do" glide path consists of portfolios 25.2/74.8 in the first year and 53.9/46.1 in the second year. The mean and standard deviation of *TAV* are 0.0777 and 0.1555 correspondingly. The optimal value of the risk-adjusted expected *TAV* is zero. The 4.5% increase over the sub-optimal \$1 payments discussed in the previous section represents the "free lunch" for objective B.

Objective C implies the following funding problem: given two \$1 payments and a risk aversion factor, minimize the asset value required to be contributed today. For this funding problem,

- the commitment out-flows are a \$1 payment at the end of the first year and a \$1 payment at the end of the second year;
- the commitment in-flow is \$*Y* contributed at the present;
- the objective is to minimize contribution \$*Y* given a risk aversion factor.

The results for various risk aversion factors are presented in *Exhibit 6*.

Exhibit 6

"Expected-To-Do" Optimal Glide Paths: $1 Payments, Minimum Asset Value											
Risk Aversion *t*	0.30	0.35	0.40	0.45	0.50	0.60	0.80	1.00	1.50	2.00	3.00
Equity Year 1	65.8%	42.7%	33.8%	28.7%	25.2%	20.9%	16.2%	13.7%	10.5%	9.0%	7.5%
Equity Year 2	100.0%	100.0%	82.0%	64.6%	53.9%	41.4%	29.2%	23.1%	15.8%	12.4%	9.2%
Expected *TAV*	0.0777	0.0753	0.0736	0.0727	0.0744	0.0813	0.1006	0.1228	0.1841	0.2515	0.4043
St Dev *TAV*	0.2590	0.2150	0.1840	0.1614	0.1488	0.1356	0.1258	0.1228	0.1227	0.1258	0.1348
Min Assets	1.872	1.886	1.897	1.906	1.914	1.928	1.953	1.977	2.038	2.102	2.245
Optimal *RAETAV*	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Here is how to read *Exhibit 6*. For risk aversion factor 0.5, the lowest contribution today is $1.914. The optimal "expected-to-do" glide path consists of portfolios 25.2/74.8 in the first year and 53.9/46.1 in the second year. The mean and standard deviation of *TAV* are 0.0744 and 0.1488 correspondingly. The optimal value of the risk-adjusted expected *TAV* is zero. Compared to the sub-optimal $2 contribution, the $0.086 (= $2.000 – $1.914) decrease discussed in the previous section represents the "free lunch" for objective C.

Some Additional Observations

- What makes a "lunch" "free"? If a "lunch" is yours for the taking, it is nominally "free." How do you do the "taking" of a nominally "free" "lunch"? You should specify the investment objectives and do the math.
- What kind of "math" should you "do"? You should develop simulation-free stochastic analysis of the key outcome variables directly connected to the investment objectives. This analysis should generate the vital measurements of the key outcome variables. These measurements are at the core of the "free-lunch"-taking process – they identify inefficiencies and ways to correct them. For example, a portfolio return (*TAV*) is the key outcome variable for the single-period funding problem discussed in this paper. The vital measurements are the mean and the variance of portfolio return (*TAV*). The calculations of these measurements are at the core of MPT.
- The funding problems discussed in this article are highly simplified. More realistic funding problems may involve longer time horizon, complete life-cycle, multiple commitment in- and out-flows, comprehensive capital market assumptions that include expected returns, risks, and correlations for all asset classes, wage and consumer inflations. Still, the methodology for the calculations of the vital measurements of the key outcome variables is readily available. Hence, the "free lunch" concepts discussed in this article are completely applicable to real life funding problems.
- There is no perfectly free lunch. The "taking" of a "free lunch" involves investing the time to learn the concepts, obtaining the analytical tools, and implementing the results. These actions

have costs, so the "lunches" discussed in this paper are only approximately "free." However, the benefits of these actions should vastly outweigh their costs.

- The magnitude of a "free lunch" is a relative quantity. It represents the difference between an optimal measurement and its sub-optimal counterpart. The magnitude of a "free lunch" can be considerable or insignificant. For instance, there is no MPT based "free lunch" for an MPT efficient portfolio.
- "Free lunch" 2.0 and 3.0 may be substantial even when there is no "free lunch" 1.0. In this paper, every portfolio is MPT efficient (meaning there is no "free lunch" 1.0), yet "free lunches" 2.0 and 3.0 are still meaningful.
- Funding problems can be evaluated from different angles. "Free lunches" 1.0, 2.0, and 3.0 identify and quantify different inefficiencies.
- The outcome variables considered in this article are based on *future values*. The selection of these valuables was primarily driven by this author's desire to present simple replicable examples that are analogous to the classic MPT. However, similar *present value* based outcome variables can be equally (or even more) valuable for the optimization of funding problems. Moreover, this author favors present value based objectives over their future value based counterparts in many cases (e.g. retirement funding).

Conclusion

The concept of the portfolio selection "free lunch" belongs to the hallmarks of finance. This article expands this concept and offers long-term investors two new generations of "free lunches." These new "free lunches" are readily available to a multitude of institutional and individual investors that includes retirement plans in general, college savings plans, foundations and endowments.

This author is optimistic that these investors will find the "free lunches" introduced in this article beneficial to their endeavors.

REFERENCES

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APPENDIX I: The Mean and Volatility of Terminal Values

Funding Problem: Investor has $2 today and a commitment to make two payments: $1 in one year and $1 in two years.

r_i is portfolio *return* in year *i*, $R_i = 1 + r_i$ is portfolio *return factor* in year *i*, *i*=1,2.

Terminal asset value (*TAV*) for this problem is defined as follows:

$$TAV = 2R_1R_2 - R_2 - 1$$

The mean of *TAV* is calculated as follows:

$$E(TAV) = 2E(R_1)E(R_2) - E(R_2) - 1$$

The volatility (standard deviation) of *TAV* is calculated as follows:

$$StDev(TAV) = \sqrt{4\left(Var(R_1)E(R_2)^2 + Var(R_2)E(R_1)^2 + Var(R_1)Var(R_2)\right) - Var(R_2)\left(4E(R_1) - 1\right)}$$

$E(R_i)$ and $Var(R_i)$, *i*=1,2, are calculated directly from the capital market assumptions.

APPENDIX II: Capital Market Assumptions

There are two asset classes: stocks and bonds. The capital market assumptions are as follows: Stocks: geometric mean 7.00%, standard deviation 16.00%. Bonds: have geometric mean 4.00%, standard deviation 5.00%. Stocks/bonds correlation: 0.2.